

# Fixed-point spectrum for group actions by affine isometries on $L_p$ -spaces.

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## Abstract

The fixed-point spectrum of a locally compact second countable group  $G$  on  $\ell_p$  is defined to be the set of  $p \geq 1$  such that every action by affine isometries of  $G$  on  $\ell_p$  admits a fixed-point. We show that this set is either empty, or is equal to a set of one of the following forms :  $[1, p_c[$ ,  $[1, p_c[ \setminus \{2\}$  for some  $1 \leq p_c \leq \infty$ , or  $[1, p_c]$ ,  $[1, p_c] \setminus \{2\}$  for some  $1 \leq p_c < \infty$ . This answers a question closely related to a conjecture of C. Drutu which asserts that the fixed-point spectrum is connected for isometric actions on  $L_p(0, 1)$ .

We also study the topological properties of the fixed-point spectrum on  $L_p(X, \mu)$  for general measure spaces  $(X, \mu)$ , and show partial results toward the conjecture for actions on  $L_p(0, 1)$ . In particular, we prove that the spectrum  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  of actions with linear part  $\pi$  is either empty, or an interval of the form  $[1, p_c]$  ( $p_c \geq 1$ ) or  $[1, \infty[$ , whenever  $\pi$  is an orthogonal representation associated to a measure-preserving ergodic action on a finite measure space  $(X, \mu)$ .

## 1 Introduction

Group actions on Banach spaces is a large topic related to many areas of mathematics : group cohomology, Kazhdan's property  $(T)$ , fixed-point properties. In [BFGM], Bader, Furman, Gelander and Monod studied group actions by isometries on Banach spaces. In particular, they gave results concerning property  $(F_{L_p(0,1)})$ , the fixed-point property for group actions by affine isometries on the space  $L_p([0, 1], \lambda)$  (abbreviated  $L_p(0, 1)$ ), where  $p \geq 1$  and  $\lambda$  denotes the Lebesgue measure. Among other results, they show that a locally compact second countable group with the fixed-point property  $(F_{L_p(0,1)})$  has the Kazhdan's property  $(T)$  when  $p \geq 1$ , and that these

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properties are equivalent when  $1 \leq p \leq 2$  (see Theorem A and Theorem 1.3 in [BFGM]).

Some groups are known to have property  $(F_{L_p(X,\mu)})$  for all  $p \geq 1$  and all standard measure space  $(X, \mu)$ : a more general result states that higher rank groups have property  $(F_{L_p(\mathcal{M})})$  for all  $p \geq 1$  and all von Neumann algebra  $\mathcal{M}$  (see Theorem B in [BFGM], Theorem 1.6 in [O], and [LS] for a stronger result). See also [M1] and [M2] where analog results are established for universal lattices  $SL_n(\mathbb{Z}[x_1, \dots, x_k])$  ( $n \geq 4$ ).

On the other hand, there exist Kazhdan groups which do not have property  $(F_{L_p(0,1)})$  for some  $p > 2$ . For instance, hyperbolic groups (and among them co-compact lattices in  $Sp(n, 1)$  have property  $(T)$ ) admit a proper action by affine isometries on  $\ell_p$ , as well as on  $L_p(0, 1)$ , for  $p$  large enough (see [Yu] and [Ni]).

The main motivation of this article is the following conjecture of C. Drutu : for all topological group  $G$ , there exists  $p_c \geq 1$  such that  $G$  has property  $(F_{L_p(0,1)})$  for  $1 \leq p < p_c$ , and  $G$  does not have property  $(F_{L_p(0,1)})$  for  $p > p_c$  (see Question 1.8 in the introduction of [CDH]). It seems that this question has its root in an older question by M. Gromov (see [G] D.6 p158). We introduce the following set and we use a similar terminology as in [No].

**Definition 1** Let  $(X, \mu)$  be a standard Borel measure space, and let  $G$  be a topological group. The set

$$\mathcal{F}_{L^\infty(X,\mu)}(G) = \{ p \geq 1 \mid G \text{ has property } (F_{L_p(X,\mu)}) \}$$

is called the fixed-point spectrum of  $G$  (for affine isometric actions) on  $L_p(X, \mu)$ -spaces.

We use the notation  $\mathcal{F}_{L^\infty(X,\mu)}(G)$  since it makes sense in a more general context, that is for actions on non-commutative  $L_p$ -spaces; in that case  $L^\infty(X, \mu)$  is replaced by a general von Neumann algebra  $\mathcal{M}$ , and the fixed-point spectrum is denoted by  $\mathcal{F}_{\mathcal{M}}(G)$ .

The conjecture of C. Drutu can be rephrased as follows : the set  $\mathcal{F}_{L^\infty(0,1)}(G)$  is connected for all group  $G$ . Our first main result is an answer to the same question, when replacing  $([0, 1], \lambda)$  by a discrete measure space. We denote by  $\ell^\infty$  the space of all bounded infinite sequences of complex numbers, and  $\ell_p$  the subspace of  $p$ -summable sequences in  $\ell^\infty$ .

**Theorem 2** *Let  $G$  be a locally compact second countable group. Then one of the following equalities holds :*

- $\mathcal{F}_{\ell^\infty}(G) = \emptyset$ ;
- $\mathcal{F}_{\ell^\infty}(G) = [1, p_c[$  for some  $1 \leq p_c \leq \infty$ ;
- $\mathcal{F}_{\ell^\infty}(G) = [1, p_c]$  for some  $1 \leq p_c < \infty$ ;
- $\mathcal{F}_{\ell^\infty}(G) = [1, p_c[ \setminus \{2\}$  and  $2 < p_c \leq \infty$ ;
- $\mathcal{F}_{\ell^\infty}(G) = [1, p_c] \setminus \{2\}$  and  $2 < p_c < \infty$ .

So the spectrum  $\mathcal{F}_{\ell^\infty}(G)$  can be empty, an interval, or the union of two disjoint intervals : we will show that these three situations can occur, depending on the group  $G$  considered.

Let us now consider the case of the fixed-point spectrum for actions on general  $L_p(X, \mu)$ -spaces. We refer to the following section 2 for more details about facts and definitions related to isometric group representations on  $L_p$ -spaces. Let  $1 \leq p < \infty$ ,  $p \neq 2$ . Let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  be an orthogonal representation. There is a natural family  $(\pi^q)_{q \geq 1}$  of orthogonal representations  $\pi^q : G \rightarrow O(L_q(X, \mu))$  associated to  $\pi^p$ , namely the conjugate representations of  $\pi^p$  by the Mazur maps  $M_{p,q}$ . Such representations  $\pi^q$ ,  $1 \leq q < \infty$ , will be called (BL) representations in this paper, in order to distinguish this class of representations from other possible classes of orthogonal representations in the case  $q = 2$  (see section 2.1 below for more details). Then we can define the fixed-point spectrum for affine isometric actions with linear parts  $(\pi^q)_{q \geq 1}$  as

$$\mathcal{F}_{L^\infty(X, \mu)}(G, (\pi^q)_{q \geq 1}) = \{ q \geq 1 \mid H^1(G, \pi^q) = \{0\} \}$$

where  $H^1(G, \pi^q)$  is the first cohomology group of  $G$  with coefficients in  $\pi^q$  (see section 2.3 for a definition). For simplicity in notations, we will denote  $\mathcal{F}_{L^\infty(X, \mu)}(G, (\pi^q)_{q \geq 1})$  by  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  for  $p \neq 2$  (but this set does not depend on  $p$ ). In the sequel, we will say that  $\pi^p$  is measure-preserving (resp. ergodic) if the associated action on  $(X, \mu)$  is measure-preserving (resp. ergodic). We say that  $\pi^p$  is positive if  $\pi^p(g)f \geq 0$  for all  $g \in G$  and all  $f \geq 0$ ,  $f \in L_p(X, \mu)$ .

The following theorem describes the form of the spectra  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  relative to measure-preserving ergodic actions.

**Theorem 3** *Let  $G$  be a second countable locally compact group. Let  $(X, \mu)$  be a finite measure space, and  $\pi : G \rightarrow O(L_p(X, \mu))$  be a (BL) measure-preserving ergodic representation. Then we have:*

- The spectrum  $\mathcal{F}_{L^\infty(X,\mu)}(G, \pi)$  is an interval or is empty.
- If  $G$  has property (T), the spectrum  $\mathcal{F}_{L^\infty(X,\mu)}(G, \pi)$  is an interval of the form  $[1, p_c]$  or  $[1, \infty[$  for some  $p_c > 2$ .

Theorem 3 is in contrast with the following well-known fact (detailed in section 5.2): if  $G$  does not have property (T), there exists some (BL) measure-preserving ergodic representation  $\rho : G \rightarrow O(L_p(X, \mu))$  such that  $\mathcal{F}_{L^\infty(X,\mu)}(G, \rho)$  is empty.

Theorems 2 and 3 give evidence toward C. Drutu's conjecture. Other partial results in that direction are proved in the paper, concerning topological properties of the fixed-point spectrum. We discuss general arguments for closeness (resp. openness) of fixed-point spectra in section 3 (resp. section 5.1). Moreover, we use recent results from [BN] about deformation of cohomology to show the connectedness of spectra  $\mathcal{F}_{L^\infty(0,1)}(G, \pi)$  under some additional assumptions on  $G$  and  $\pi$ .

The paper is organized as follows. In section 2, we recall general facts and results we will need about isometric group actions on  $L_p$ -spaces. In section 3, we show general results concerning the closeness of the fixed-point spectrum. Section 4 is devoted to the proof of Theorem 2. In section 5, we prove various partial results toward a proof for the conjecture of C. Drutu concerning the case of  $L_p(0, 1)$ .

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## 2 Isometric group actions on $L_p(X, \mu)$ -spaces

In this section, we recall some general definitions and properties of linear and affine isometric actions on  $L_p(X, \mu)$ -spaces. Let  $G$  be a topological group, and  $(X, \mu)$  be a standard Borel measure space.

### 2.1 Orthogonal representations on $L_p(X, \mu)$

Let  $1 \leq p < \infty$ ,  $p \neq 2$ , and denote by  $O(L_p(X, \mu))$  the group of bijective linear isometries of the space  $L_p(X, \mu)$ . By a theorem of Banach and Lam-

perti, elements in  $O(L_p(X, \mu))$  are the linear maps  $U : L_p(X, \mu) \rightarrow L_p(X, \mu)$  described as follows :

$$Uf(x) = h(x) \left( \frac{d\varphi * \mu}{d\mu}(x) \right)^{1/p} f(\varphi(x)) \quad (*)$$

where  $h : X \rightarrow \mathbb{C}$  is a measurable function of modulus one, and  $\varphi : X \rightarrow X$  is a  $\mu$ -class-preserving bijective transformation. In the particular case where  $(X, \mu)$  is a discrete measure space, then  $\varphi$  is a permutation of the countable set  $X$ .

An orthogonal representation  $\pi$  of the group  $G$  on  $L_p(X, \mu)$  is a group homomorphism  $\pi : G \rightarrow O(L_p(X, \mu))$  such that the maps  $g \mapsto \pi(g)f$  are continuous on  $G$  for all  $f \in L_p(X, \mu)$ . Then the set of  $\pi(G)$ -invariant vectors is denoted by  $L_p(X, \mu)^{\pi(G)}$ . When  $p > 1$ , we have the following  $\pi(G)$ -invariant decomposition of the space  $L_p(X, \mu)$  :

$$L_p(X, \mu) = L_p^{\pi(G)} \oplus L'_p(\pi)$$

where  $L'_p(\pi) = \{ f \in L_p \mid \forall h \in L_p^{\pi^*(G)}, \langle f, h \rangle = 0 \}$  is the annihilator of the  $G$ -invariant vectors for the contragradient representation  $\pi^* : G \rightarrow O(L_{p'})$  of  $\pi$ ,  $p' = p/(p-1)$  (see Proposition 2.6 and section 2.c for more details in [BFGM]). Then one can consider the restriction  $\pi' : G \rightarrow O(L'_p(\pi))$ . When  $p = 1$ , the analog of the previous representation is the representation  $\pi' : G \rightarrow O(L_1/L_1^{\pi(G)})$ , obtained from  $\pi$  by composing with the quotient projection. If the context is clear, we will keep the notation  $\pi$  in place of  $\pi'$  in the sequel.

When  $p = 2$ , the isometry group  $O(L_2(X, \mu))$  contains much more elements than those described by formulae (\*) in the Banach-Lamperti theorem. Orthogonal representations given by formulae (\*) will play a central role in our study. To distinguish them from the others (when  $p = 2$ ), they will be called (BL) orthogonal representations in the sequel.

Let  $\pi : G \rightarrow O(L_p(X, \mu))$  be a (BL) orthogonal representation. We will say that  $\pi$  is measure-preserving if the corresponding action of  $G$  on  $(X, \mu)$  is  $\mu$ -preserving, equivalently if the associated Radon-Nikodym derivative is equal to 1 almost everywhere. By Banach's description of the isometries of  $\ell_p$ , all (BL) representations on the space  $\ell_p$  are measure-preserving (see section 2 in [BO] for details).

## 2.2 The Mazur map

A very useful tool to study  $L_p$ -spaces and their representations is the Mazur map. Here we recall the definition and some well-known properties of this map. The proof of the basic properties of the Mazur map can be found in Chapter 9.1 of [BL].

Let  $1 \leq p, q \leq \infty$ . The map

$$M_{p,q} : L_p(X, \mu) \rightarrow L_q(X, \mu) \\ f = sg(f)|f| \mapsto sg(f)|f|^{p/q}$$

is called the Mazur map. It induces a uniformly continuous homeomorphism between the unit spheres of  $L_p(X, \mu)$  and  $L_q(X, \mu)$ . More precisely, it satisfies the following estimates for  $1 \leq p < q < \infty$  :

$$\|M_{p,q}(x) - M_{p,q}(y)\|_q \leq C_{p,q} \|x - y\|_p^{\theta_{p,q}} \quad (1)$$

for all  $x, y \in S(L_p(X, \mu))$ , where the constant  $C_{p,q}$  only depends on  $p, q$  (the opposite inequalities hold when  $q < p$ ), and  $\theta_{p,q} = \min(1, \frac{p}{q})$ .

Here we make an important remark for our proofs. Assume that we have a sequence  $(p_n)_n$  of real numbers in  $[1, \infty[$  which converges to  $p \geq 1$ . Then we can find a constant  $C > 0$  (independent of  $n$ ) such that inequalities (1) hold for all Mazur maps  $M_{p,p_n}, M_{p_n,p}$  when replacing  $C_{p,p_n}$  and  $C_{p_n,p}$  by  $C$ . This fact can be deduced from the proof of Theorem 9.1 in [BL].

Let  $L_0(X, \mu)$  be the closure of  $L^\infty(X, \mu)$  with respect to the measure topology. Then it is well-known that  $L_0(X, \mu)$  contains  $L_p(X, \mu)$  for all  $p \geq 1$ . Moreover, the following inequalities hold when  $p < q$  :

$$\|M_{p,q}(x) - M_{p,q}(y)\|_q \leq \|x - y\|_p^{p/q} \text{ for all } x, y \in L_0(X, \mu), x, y \geq 0 \quad (2).$$

The previous inequalities for the Mazur maps hold in a much more general context, that is for Mazur maps on non-commutative  $L_p$ -spaces : we refer to [R] for complete proofs in this general context.

Now assume  $p \neq 2$ , and let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  be an orthogonal representation of  $G$  on  $L_p(X, \mu)$ . Then the formulae

$$\pi^q(g)x = (M_{p,q} \circ \pi^p(g)) \circ M_{q,p}x \text{ for all } g \in G, x \in L_q(X, \mu) \quad (3)$$

define an orthogonal representation  $\pi^q : G \rightarrow O(L_q(X, \mu))$ , by the description of  $O(L_p(X, \mu))$  of Banach-Lamperti theorem.

A sequence  $(f_n)_n$  in a Banach space  $F$  is said to be a sequence of almost invariant vectors for an orthogonal representation  $\pi : G \rightarrow O(F)$  if  $\|f_n\|_F = 1$  and

$$\lim_n \|\pi(g)f_n - f_n\|_F = 0 \text{ uniformly on compact subsets of } G.$$

It is well known that a group  $G$  admits a sequence of almost invariant vectors for the representation  $(\pi^p)'$  if and only if it admits a sequence of almost invariant vectors for  $(\pi^q)'$ , whenever  $1 \leq p, q < \infty$  (see section 4.a in [BFGM]).

### 2.3 Affine isometric actions on $L_p(X, \mu)$

Denote by  $\text{Isom}(L_p(X, \mu))$  the group of affine bijective isometries of  $L_p(X, \mu)$ . An affine isometric action of  $G$  on  $L_p(X, \mu)$  is a group homomorphism  $\alpha : G \rightarrow \text{Isom}(L_p(X, \mu))$  such that the maps  $g \mapsto \alpha(g)f$  are continuous for all  $f \in L_p(X, \mu)$ . Then we have

$$\alpha(g)f = \pi(g)f + b(g) \text{ for all } g \in G, f \in L_p(X, \mu)$$

where  $\pi : G \rightarrow O(L_p(X, \mu))$  is an orthogonal representation, and  $b : G \rightarrow L_p(X, \mu)$  is a 1-cocycle associated to  $\pi$ , that is a continuous map satisfying the following relations :

$$b(gh) = b(g) + \pi(g)b(h) \text{ for all } g, h \in G.$$

Given an orthogonal representation  $\pi$  and a cocycle associated to  $\pi$ , we will sometimes use the notation  $\alpha = (\pi, b)$  to denote the affine representation whose linear part is  $\pi$ , and translation part is  $b$ . We denote by  $H^1(G, \pi)$  the first cohomology group with coefficients in  $\pi$ , that is the quotient of the space of 1-cocycles associated to  $\pi$ , by the subspace of 1-coboundaries (a coboundary is a cocycle of the form  $b(g) = f - \pi(g)f$  for some  $f \in L_p(X, \mu)$ ).

We recall that an affine isometric action of  $G$  on  $L_p(X, \mu)$  has a fixed-point if and only if the associated cocycle  $b : G \rightarrow L_p(X, \mu)$  is a bounded map (see Lemma 2.14 in [BFGM] for  $p > 1$ , and [BGM] for  $p = 1$ ). The action is said to be proper if  $\lim_{g \rightarrow \infty} \|b(g)\|_p = \infty$ . A topological group  $G$  is said to have the fixed-point property ( $F_{L_p(X, \mu)}$ ) if every action by affine isometries of  $G$  on  $L_p(X, \mu)$  admits a fixed-point, that is  $H^1(G, \pi) = \{0\}$  for all orthogonal representation  $\pi : G \rightarrow O(L_p(X, \mu))$ .

A locally compact second countable group  $G$  is said to have property  $(T_{L_p(X,\mu)})$  if for any orthogonal representation  $\pi : G \rightarrow O(L_p(X,\mu))$ , the restriction of  $\pi$  on  $L'_p(\pi)$  or  $L_1/L_1^{\pi(G)}$ , has no sequence of almost invariant vectors. Recall the following fact due to Guichardet (see [BFGM] section 3.a for a proof) : if an orthogonal representation  $\pi : G \rightarrow O(L_p(X,\mu))$  of a second countable group  $G$  has a sequence of almost invariant vectors for  $(\pi)'$ , then  $H^1(G, \pi) \neq \{0\}$ . We will use the latter argument in the sequel, and also an important consequence of this: property  $(F_{L_p(X,\mu)})$  implies property  $(T_{L_p(X,\mu)})$  (these results hold in a much more general context: see Theorem 1.3 in [BFGM]).

In the case where the group  $G$  is second countable and generated by a compact subset  $Q \subset G$ , an orthogonal representation  $\pi : G \rightarrow O(L_p(X,\mu))$  without any sequence of almost invariant vectors in  $L_p(X,\mu)'$  satisfies the following condition : there exists  $\epsilon > 0$  such that for all  $x \in L_p(X,\mu)'$ ,

$$\epsilon \|x\|_p \leq \|x - \pi(g)x\|_p \text{ for some } g \in Q.$$

When  $p = 2$  and the previous condition holds for all orthogonal representations  $\pi : G \rightarrow O(L_2(X,\mu))$ , the pair  $(Q, \epsilon)$  is called a Kazhdan pair.

### 3 About the closeness of the fixed-point spectrum

A possible way to show that the fixed-point spectrum  $\mathcal{F}_{L^\infty(X,\mu)}$  is an interval containing 1, is to study closeness and openness properties of this set in  $[1, \infty[$ . In this section, we show closeness results for  $\mathcal{F}_{L^\infty(X,\mu)}(G, \pi)$ . To prove this, we use a limit-version of a well-known fact about almost invariant vectors for (BL) representations, which is interesting in its own right (Proposition 5 below).

We begin with some general facts concerning isometric actions on  $L_p$ -spaces.

**Lemma 4** *Let  $1 \leq p, p_n < \infty$ ,  $p_n \neq 1$ , be such that  $\lim_n p_n = p$ . Let  $G$  be a topological group. Let  $(X, \mu)$  be a standard Borel measure space. Let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  be a (BL) orthogonal representation. Then there exists  $C > 0$  and  $N$  such that for all  $n \geq N$ , we have*

$$d(M_{p_n,p}f, L_p(X, \mu)^{\pi^p(G)}) \geq C \text{ for all } f \in S(L_{p_n}(X, \mu)'(\pi^{p_n})).$$



**Proof** For  $f \in S(L'_{p_n}(\pi^{p_n}))$ , the following inequalities hold (see Proposition 3.5 in [O]) :

$$d(f, L_{p_n}^{\pi^{p_n}(G)}) \geq 1/2 \text{ for all } n.$$

Notice that  $L_p^{\pi^p(G)} = M_{p_n,p}(L_{p_n}^{\pi^{p_n}(G)})$ . If the result does not hold, there exist  $f_n \in S(L'_{p_n}(\pi^{p_n}))$  and  $a_n \in L_{p_n}^{\pi^{p_n}(G)}$  such that

$$\lim_n \|M_{p_n,p}f_n - M_{p_n,p}a_n\|_{p_n} = 0.$$

Recall that the constant  $C$  in estimates (1) from section 2.2, applied to  $M_{p_n,p}$ , can be made independant of  $n$  since  $\lim_n p_n = p$ . Hence the latter inequalities and the convergence to 0 above contradict the fact that

$$d(f_n, L_{p_n}^{\pi^{p_n}(G)}) \geq 1/2 \text{ for all } n.$$

■

The following proposition is a limit-version of a well-known fact about almost invariant vectors for (BL) representations (see the end of section 2.2).

**Proposition 5** *Let  $1 \leq p, p_n < \infty, p_n \neq 1$ , be such that  $\lim_n p_n = p$ . Let  $G$  be a second countable topological group. Let  $(X, \mu)$  be a standard measure space. Let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  be a (BL) orthogonal representation. Assume that, for all  $n$ , there exists  $f_n \in S(L_{p_n}(X, \mu)'(\pi^{p_n}))$  such that*

$$\limsup_n \sup_{g \in Q} \|f_n - \pi^{p_n}(g)f_n\|_{p_n} = 0$$

*for all compact subsets  $Q \subset G$ . Then there exists a sequence of almost invariant vectors for  $\pi^p$  in  $L_p(X, \mu)'(\pi^p)$ .*

**Proof** Let  $Q \subset G$  be a compact set. Define  $h_n = M_{p_n,p}f_n \in S(L_p(X, \mu))$  for  $n \in \mathbb{N}$ . As in the previous lemma, we can choose  $C$  for the estimates of the Mazur maps  $M_{p_n,p}$ , independant of  $n$ . Then we have

$$\limsup_n \sup_{g \in Q} \|h_n - \pi^p(g)h_n\|_p = 0. \quad (*)$$

Now by Lemma 4, there exists  $C' > 0$  such that  $d(h_n, L_p^{\pi^p(G)}) \geq C'$  holds for all  $n$  large enough.

In the case where  $p > 1$ , consider  $v_n$  the projection of  $h_n$  on the complement subspace  $L'_p(\pi^p)$ . Since  $\|v_n\|_p \geq C'$  for  $n$  large enough, the uniform convergence (\*) on compact subsets holds when replacing  $h_n$  by  $v_n/\|v_n\|_p$ . Hence

$(v_n/\|v_n\|_p)_n$  is a sequence of almost invariant vectors for  $\pi^p$  with values in  $L'_p(\pi^p)$ .

For  $p = 1$ , we consider  $v_n$  the projection of  $h_n$  on the quotient space  $F = L_1/L_1^{\pi^p(G)}$ . Then  $\|v_n\|_F \geq C'$ , and the sequence  $v_n/\|v_n\|_F$  is a sequence of almost invariant vectors for the representation  $(\pi^1)' : G \rightarrow O(F)$ .  
■

Now we show a closeness property for some sets  $\mathcal{F}_{L^\infty(X,\mu)}(G, \pi)$ . Our proof requires two important assumptions : the monotonicity of  $(L_p(X, \mu))_p$  for the inclusion, and the representation  $\pi$  to be measure-preserving.

**Proposition 6** *Let  $1 \leq p, p_n < \infty$ ,  $p_n \neq 1$ , be such that  $\lim_n p_n = p$ . Assume that  $L_p(X, \mu) \subset L_{p_n}(X, \mu)$  for all  $n$ , and that  $\lim_n \|f\|_{p_n} = \|f\|_p$  for all  $f \in L_p(X, \mu)$ . Let  $G$  be a topological group. Let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  be a (BL) measure-preserving orthogonal representation. Assume also that*

$$H^1(G, \pi^{p_n}) = \{0\}.$$

*Then we have*

$$H^1(G, \pi^p) = \{0\}.$$

**Proof** Let  $b : G \rightarrow L_p(X, \mu)$  be a cocycle associated to  $\pi^p$ . Let  $n \in \mathbb{N}$ . Since by assumption  $L_p(X, \mu) \subset L_{p_n}(X, \mu)$  and  $\pi^p$  is measure-preserving,  $b$  defines also a cocycle for the representation  $\pi^{p_n}$ . Then there exists  $f_n \in L_{p_n}(X, \mu)$  such that

$$b(g) = f_n - \pi^{p_n}(g)f_n \text{ for all } g \in G.$$

Notice that we can assume that  $f_n \in L'_{p_n}(\pi^{p_n})$  for all  $n$ , without loss of generality. We claim that  $(\|f_n\|_{p_n})_n$  is bounded. To prove this, we assume the contrary and we show that  $(\pi^p)'$  has a sequence of almost invariant vectors, which is not possible by assumption. Take a subsequence of  $(\|f_n\|_{p_n})_n$  which tends to  $\infty$  (and use the same notation for the subsequence). Let  $Q \subset G$  be a compact subset. Since  $\lim_n \|b(g)\|_{p_n} = \|b(g)\|_p$  for all  $g \in G$ , for  $n$  large enough we have

$$\begin{aligned} \sup_{g \in Q} \left\| \frac{f_n}{\|f_n\|_{p_n}} - \pi^{p_n}(g) \frac{f_n}{\|f_n\|_{p_n}} \right\|_{p_n} &= \frac{\sup_{g \in Q} \|b(g)\|_{p_n}}{\|f_n\|_{p_n}} \\ &\leq 2 \frac{\sup_{g \in Q} \|b(g)\|_p}{\|f_n\|_{p_n}} \end{aligned}$$

and the right-hand side of the inequality tends to 0 as  $n$  tends to  $\infty$ . By Proposition 5, there exists a sequence of almost invariant vectors for  $(\pi^p)'$ . Hence by a general argument recalled in section 2.2, the representation  $(\pi^{p_n})'$  admits a sequence of almost invariant vectors as well. This contradicts the vanishing of  $H^1(G, \pi^{p_n})$ .

Then there exists  $C > 0$  such that  $\|f_n\|_{p_n} \leq C$  for all  $n \in \mathbb{N}$ . Let  $g \in G$ . We have

$$\|b(g)\|_p = \lim_n \|b(g)\|_{p_n} \leq 2C.$$

Hence the cocycle  $b : G \rightarrow \ell_p$  is bounded in  $\ell_p$ . ■

## 4 The fixed-point spectrum for actions on $\ell_p$

This section is divided in two parts. In the first part, we prove Theorem 2. Then in the second part, we discuss property  $(F_{\ell_p})$  and exhibit examples of groups  $G$  for each form of possible fixed-point spectra  $\mathcal{F}_{\ell^\infty}(G)$  listed in the statement of Theorem 2.

### 4.1 Proof of Theorem 2

Theorem 2 is a direct consequence of Proposition 7, which we state and prove after a few useful remarks.

Let  $\pi : G \rightarrow O(\ell_p)$  be a  $(BL)$  orthogonal representation. Write  $\ell_p = \ell_p(X)$  and decompose  $X = X' + X''$  where  $X'$  is the union of the finite orbits of the  $G$ -action, and  $X''$  the union of the infinite orbits. In the sequel, we will make use of the decomposition of  $\pi$  as  $\pi = \pi' \oplus \pi''$  with respect to the decomposition  $\ell_p(X) = \ell_p(X') \oplus \ell_p(X'')$ .

We make the following observation :

$$H^1(G, \pi) = \{0\} \Leftrightarrow H^1(G, \pi') = \{0\} \text{ and } H^1(G, \pi'') = \{0\}.$$

Hence  $\mathcal{F}_{\ell^\infty}(G, \pi)$  is an interval of the form  $[1, p_c]$  or  $[1, p_c[$  for some  $p_c \geq 1$  if and only if the two following conditions hold:

- (i)  $\mathcal{F}_{\ell^\infty}(G, \pi')$  is an interval  $[1, p'_c]$  or  $[1, p'_c[$  for some  $p'_c \geq 1$ ;
- (ii)  $\mathcal{F}_{\ell^\infty}(G, \pi'')$  is an interval  $[1, p''_c]$  or  $[1, p''_c[$  for some  $p''_c \geq 1$ .

In the sequel, we will denote by  $|\pi|$  the positive part of the representation  $\pi$ , defined by  $|\pi|(g)f = |\pi(g)f|$  for all  $g \in G$ , and all  $f \in \ell_p$ ,  $f \geq 0$ .

For technical reasons, we are not able to prove that  $\mathcal{F}_{\ell^\infty}(G, \pi)$  is an interval for all representations  $\pi$  (but for positive representations, we do). Nevertheless, the following Proposition 7 shows a very close result to the connectedness of all  $\mathcal{F}_{\ell^\infty}(G, \pi)$ , and this result clearly implies Theorem 2. Notice that when a group  $G$  is not compactly generated, then  $\mathcal{F}_{\ell^\infty}(G)$  is empty: for any  $p \geq 1$ , such a group does not have property  $(T_{\ell_p})$  (see [BO]), nor property  $(F_{\ell_p})$  by well-known results.

**Proposition 7** *Let  $G$  be a second countable group generated by a compact set  $Q$ . Let  $\pi : G \rightarrow O(\ell_p)$  be a (BL) orthogonal representation. Let  $\pi', \pi''$  be defined as in the previous discussion. Then we have:*

- (i)  $\mathcal{F}_{\ell^\infty}(G, \pi') = [1, p'_c]$  or  $[1, p'_c[$ ;
- (ii)  $\mathcal{F}_{\ell^\infty}(G, \pi'') \cap \mathcal{F}_{\ell^\infty}(G, |\pi''|) = [1, p''_c]$  or  $[1, p''_c[$ .

Before giving the proof of Proposition 7, we start with a simple lemma.

**Lemma 8** *Let  $1 \leq p < q$ . Let  $\pi^p : G \rightarrow O(\ell_p)$  be a (BL) orthogonal representation without non-zero invariant vectors. Assume that*

$$H^1(G, \pi^q) = \{0\} \text{ and } H^1(G, |\pi^p|) = \{0\}.$$

*Then we also have*

$$H^1(G, \pi^p) = \{0\}.$$

**Proof** Let  $b : G \rightarrow \ell_p$  be a cocycle for  $\pi$ . By assumption, there exists  $x \in \ell_q$  such that

$$b(g) = x - \pi(g)x \text{ for all } g \in G.$$

For all  $a, b \in \mathbb{C}$ , we have

$$||a| - |b|| \leq |a - b|,$$

and the following inequalities follow for all  $g \in G$  :

$$||x| - |\pi^p|(g)|x||_p \leq ||b(g)||_p.$$

Hence the left handside defines a cocycle for  $|\pi^p|$  with values in  $\ell_p$ . By assumption, there exists  $y \in \ell_p$  such that

$$|x| - |\pi^p|(g)|x| = y - |\pi^p|(g)y \text{ for all } g \in G.$$

Since  $\ell_p = \ell'_p(\pi^p)$ , we have  $|x| = y$ , and then  $x \in \ell_p$ . ■

Now we are able to prove Proposition 7.

**Proof of Proposition 7** (i) We show that for  $1 < p < q$ ,  $H^1(G, (\pi^q)') = \{0\}$  implies  $H^1(G, (\pi^p)') = \{0\}$ . The case  $p = 1$  is a straightforward consequence of Proposition 6.

Let  $b : G \rightarrow \ell_p(X')$  be a cocycle associated to  $\pi'$ . By assumption, there exists  $x \in \ell_q$  such that

$$b(g) = x - \pi(g)x \text{ for all } g \in G.$$

Notice that we can assume that  $x \in \ell'_q$  without loss of generality. We decompose  $X' = \sqcup_{i \geq 0} X_i$  in (finite) orbits, and write  $x = \oplus_i x_i$  in  $\ell_q(X') = \oplus_i \ell_q(X_i)$ . From the definition of the complement  $\ell'_q$  (see section 2.1), it is clear that  $x_i \in \ell_q(X_i)'$  for all  $i$ .

Since  $\pi'$  has no sequence of almost invariant vectors, there exists  $\epsilon > 0$  such that

$$\epsilon \|f\|_p \leq \|f - \pi(g)f\|_p \text{ for all } f \in \ell'_p(\pi'), g \in Q.$$

Define  $u_n = \oplus_{i \leq n} x_i \in \ell'_q$ . Then we have for all  $g \in Q$ , and all  $n$ ,

$$\begin{aligned} \epsilon \|u_n\|_p &\leq \|u_n - \pi(g)u_n\|_p \\ &\leq \|b(g)\|_p \\ &\leq M \end{aligned}$$

where we denote  $M = \sup_{g \in Q} \|b(g)\|_p$ . Hence the sequence  $(u_n)_n$  is bounded in  $\ell_p$  by  $M/\epsilon$ .

On the other hand, we have

$$\lim_n \|u_n - \pi(g)u_n - b(g)\|_p = 0 \text{ for all } g \in G.$$

Then for  $g \in G$ , we have

$$\begin{aligned} \|b(g)\|_p &= \lim_n \|u_n - \pi(g)u_n\|_p \\ &\leq 2M/\epsilon. \end{aligned}$$

So the cocycle  $b$  is bounded in  $\ell_p$ , and the proposition is proved.

(ii) Let  $1 \leq p < q$ . We assume that  $q \in \mathcal{F}_{\ell^\infty}(G, \pi'') \cap \mathcal{F}_{\ell^\infty}(G, |\pi''|)$ . Then we need to show that  $p \in \mathcal{F}_{\ell^\infty}(G, \pi'') \cap \mathcal{F}_{\ell^\infty}(G, |\pi''|)$ . By lemma 8, it is sufficient to show that  $H^1(G, |(\pi^p)'|) = \{0\}$ .

Let  $b : G \rightarrow \ell_p(X'')$  be a cocycle associated to  $|(\pi^p)''|$ . By assumption, there exists  $x \in \ell_q$  such that

$$b(g) = x - |\pi''|(g)x \text{ for all } g \in G.$$

On the one hand, we have for all  $g \in G$ ,

$$||x| - |\pi''|(g)|x||_p \leq ||x - |\pi''|(g)x||_p.$$

On the other hand, estimates (2) for the Mazur maps in section 2.2 imply the following inequalities for all  $g \in G$  :

$$||M_{p,q}(|x|) - |\pi''|(g)M_{p,q}(|x|)||_q \leq ||x| - |\pi''|(g)|x||_p.$$

Hence the formulae  $M_{p,q}(|x|) - |\pi''|(g)M_{p,q}(|x|)$ ,  $g \in G$ , define a cocycle associated to  $|\pi|$  with values in  $\ell_q$ . By assumption, there exists  $y \in \ell_q$  such that

$$M_{p,q}(|x|) - |\pi''|(g)M_{p,q}(|x|) = y - |\pi''|(g)y \text{ for all } g \in G.$$

Since  $X''$  has only infinite orbits for the  $G$ -action, there is no non-zero  $\pi''(G)$ -invariant vector in  $\ell_q$ . Hence we have  $|x| = M_{q,p}y$ . It follows that  $x \in \ell_p$  and the proposition is proved. ■

As a particular case of the previous proofs, we have the following result.

**Corollary 9** *Let  $G$  be a second countable group, and  $p \geq 1$ . Let  $\pi : G \rightarrow O(\ell_p)$  be a (BL) orthogonal positive representation. Then  $\mathcal{F}_{\ell^\infty}(G, \pi)$  is empty, or is an interval of the form  $[1, p_c[$  ( $p_c \leq \infty$ ) or  $[1, p_c]$  ( $p_c < \infty$ ).*

## 4.2 More about property $(F_{\ell_p})$

Now we give examples of groups  $G$  for which the fixed-point spectrum  $\mathcal{F}_{\ell^\infty}(G)$  is the union of two intervals  $[1, p_c[ \setminus \{2\}$  ( $p_c > 2$ ). We recall the following well-known relationships between properties  $(T)$ ,  $(T_{\ell_p})$  and  $(F_{\ell_p})$ :

$$\begin{aligned} (F_{\ell_p}) &\Rightarrow (T_{\ell_p}) && \text{for all } p \geq 1, \\ (T) &\Rightarrow (F_{\ell_p}) \Rightarrow (T_{\ell_p}) && \text{for } 1 \leq p \leq 2. \end{aligned}$$

Property  $(T_{\ell_p})$  was studied in [BO]. In particular, it was shown that in the class of connected groups, groups having property  $(T_{\ell_p})$  are the ones with compact abelianization (see Corollary 3 in [BO]). We now show that the latter groups also have property  $(F_{\ell_p})$ .

**Theorem 10** *Let  $G$  be a locally compact second countable group. Assume that  $G$  is connected. Then the following assertions are equivalent :*

- (i)  $G$  has property  $(F_{\ell_p})$ ;
- (ii)  $G$  has property  $(T_{\ell_p})$ ;
- (iii) the abelianised group  $G/\overline{[G, G]}$  is compact.

**Proof** We only have to show the implication (iii)  $\Rightarrow$  (i). Since  $G$  is connected, the orbits of a continuous action of  $G$  on a countable infinite discrete set  $X$  are singletons. Hence a (BL) representation on  $\ell_p(X)$  is a direct sum of continuous unitary characters. So we have to show that  $H^1(G, \pi) = \{0\}$  for every orthogonal representation  $\pi : G \rightarrow O(\ell_p)$  of the form

$$\pi = \bigoplus_{i \in I} \chi_i$$

where every  $\chi_i$  is a continuous character on  $G$ . Let  $\pi = \bigoplus_{i \in I} \chi_i$  be such a representation. Let  $H = \bigcap_{i \in I} \text{Ker}(\chi_i)$  be the kernel of the homomorphism

$$\begin{aligned} \varphi : G &\rightarrow \prod_{i \in I} S^1 \\ g &\rightarrow (\chi_i(g))_{i \in I}. \end{aligned}$$

Denote by  $N = \overline{[G, G]}$  and  $p : G \rightarrow G/N$  the quotient projection. Notice that, since  $N \subset H$ ,  $\pi(n) = id$  for all  $n \in N$  and  $\ell_p^{\pi(N)} = \ell_p$ . So  $\pi$  factorizes through  $G/N$  as  $\pi = \rho \circ p$ . As a consequence of the Hoschild-Serre spectral sequence, we have the following exact sequence :

$$H^1(G/N, \rho) \rightarrow H^1(G, \pi) \rightarrow H^1(N, 1_N)^{G/N}.$$

To finish the proof, it suffices to notice that  $H^1(G/N, \rho) = \{0\}$  and  $H^1(N, 1_N) = \{0\}$ . Indeed, since  $G/N$  is compact, we have  $H^1(G/N, \rho) = \{0\}$ . Moreover,  $H^1(N, 1_N) = \text{Hom}(N, \ell_p) = \{0\}$  since  $\ell_p$  is commutative and  $N = \overline{[G, G]}$ . ■

**Remark 11** (i) The exact sequence in the previous proof was already used in [BMV] to obtain results concerning the cohomology (and the reduced cohomology) associated to the regular representation. In particular, it is shown that the reduced  $\ell_p$ -cohomology  $\overline{H}_p^1(\Gamma, \lambda_\Gamma)$  vanishes if and only if  $p \leq e(\Gamma)$  for some lattices  $\Gamma$  in rank one groups and some critical exponent  $e(\Gamma)$  explicitly defined (see Theorem 2 and the discussion which follows its statement). Such a result suggests that the critical point  $p_c$  should belong

to the fixed-point spectrum  $\mathcal{F}_{\ell^\infty}(G, \pi)$ , or at least to its analog defined via reduced cohomology.

(ii) Other examples which suggest that the set  $\mathcal{F}_{\ell^\infty}(G, \pi)$  is closed can be found in [B] ( see Remark 4 in section 1.6, and Remark 3 in section 2.4 ).

**Question 12** Does  $p_c$  always belong to  $\mathcal{F}_{\ell^\infty}(G, \pi)$  when the fixed-point spectrum is non-empty and  $\pi$  is a positive (BL) representation ?

The following examples show that the the fixed-point spectrum can have one or two connected components when it is not empty.

**Examples 13** (i) For instance, the group  $SL_2(\mathbb{R})$  does not have property (T), but has property  $(F_{\ell_p})$  for all  $p \neq 2$  by the previous theorem. So we have  $\mathcal{F}_{\ell^\infty}(SL_2(\mathbb{R})) = [1, \infty[ \setminus \{2\}$ .

(ii) The group  $SL_2(\mathbb{Q}_l)$  (where  $\mathbb{Q}_l$  is the field of  $l$ -adic numbers) has property  $(T_{\ell_p})$  (see Exemple 9 in [BO]). On the other hand,  $SL_2(\mathbb{Q}_l)$  is known to act on a tree without fixed point. Hence it does not have property  $(F_{\ell_p})$  for any  $p \geq 1$  (see [BHV] section 2.3 p.87 for instance). So property  $(T_{\ell_p})$  is stricly weaker than property  $(F_{\ell_p})$ . Moreover,  $G = SL_2(\mathbb{Q}_l)$  is an example of a group such that  $\mathcal{F}_{\ell^\infty}(G)$  is empty.

(iii) Any group  $G$  with property (T) has a spectrum  $\mathcal{F}_{\ell^\infty}(G)$  of the form  $[1, p_c[$  or  $[1, p_c]$  for some  $p_c > 2$ .

**Question 14** Does there exist a second countable non-Kazhdan totally disconnected group which has property  $(F_{\ell_p})$  for some (all)  $1 < p < 2$  ?

## 5 Fixed-point spectrum for actions on $L_p(X, \mu)$ associated to non-atomic measure spaces $(X, \mu)$

This section is devoted to the study of connectedness properties for  $\mathcal{F}_{L^\infty(X, \mu)}(G)$  for general measure spaces  $(X, \mu)$ . We give partial results concerning C. Drutu's conjecture asserting that  $\mathcal{F}_{L^\infty(0,1)}(G)$  is connected.

### 5.1 Openness of the spectrum $\mathcal{F}_{L^\infty}(G)$

The openness of  $\mathcal{F}_{L^\infty(0,1)}(G)$  at 2 is a well-known fact, due to Fisher and Margulis. Their argument uses a limit action on an ultraproduct of  $L_p$ -spaces, which we recall briefly in this section. Then we explain how it is used to show some openness properties relative to the fixed-point spectrum  $\mathcal{F}_{\ell^\infty(0,1)}(G)$ . We refer to the survey [H] for more details on ultraproducts of



Banach spaces.

We now recall the construction of the ultraproduct space of  $L_p$ -spaces in the context which is relevant for our purpose : we will use the ultraproduct of  $L_{p_n}(X, \mu)$ -spaces over the same measure space  $(X, \mu)$ , but such that  $(p_n)_n$  is a sequence of real numbers converging to  $p \geq 1$ .

Let  $1 \leq p, p_n < \infty$  be real numbers such that  $\lim_n p_n = p$ . Let  $(X, \mu)$  be a measure space. Fix a non-principal ultrafilter  $\mathcal{U}$  on  $\mathbb{N}$ . We recall the construction of the ultraproduct (affine) space of the  $L_{p_n}(X, \mu)$ -spaces with marked points  $x_n \in L_{p_n}(X, \mu)$ . The latter affine space is defined as

$$(x_n)_n + (L_{p_n})_{\mathcal{U}} = (x_n)_n + (\prod_n L_{p_n})_{\infty} / \mathcal{N}$$

where

$$(\prod_n L_{p_n})_{\infty} = \{ (f_n)_n \mid \sup_n \|f_n\|_{p_n} < \infty \},$$

and

$$\mathcal{N} = \{ (f_n)_n \in (\prod_n L_{p_n})_{\infty} \mid \|(f_n)_n\|_{\mathcal{U}} = 0 \} \text{ for } \|(f_n)_n\|_{\mathcal{U}} := \lim_{n, \mathcal{U}} \|f_n\|_{p_n}.$$

An ultraproduct of Banach lattices is still a Banach lattice. Moreover, the norm  $\|\cdot\|$  on  $(L_{p_n})_{\mathcal{U}}$  is clearly  $p$ -additive since  $\lim_n p_n = p$ . Hence by the generalized Kakutani representation theorem,  $(L_{p_n})_{\mathcal{U}}$  is isometrically isomorphic to  $L_p(Y, \nu)$  for some measure space  $(Y, \nu)$ .

Let  $G$  be a locally compact group generated by a compact subset  $Q \subset G$ . Let  $\alpha_n = (\pi_n, b_n)$  be affine isometric actions of  $G$  on the spaces  $L_{p_n}(X, \mu)$ . In the sequel, the diameter of a set  $X$  is denoted by  $\text{diam}(X)$ . Under the assumption that  $\text{diam}(\alpha_n(Q)x_n)$  is bounded for all  $n$ , we can define an isometric affine action  $\alpha$  on the affine space  $(x_n)_n + (L_{p_n})_{\mathcal{U}}$  by the following formulae :

$$\alpha(g)(x_n) + (f_n)_{\mathcal{U}} = (x_n)_n + (\alpha_n(g)x_n - x_n + \pi_n(g)f_n)_n$$

for all  $(f_n)_{\mathcal{U}} \in (L_{p_n})_{\mathcal{U}}$  and all  $g \in G$ . This is not clear that the limit action needs to be continuous when  $G$  is only assumed to be locally compact. We will discuss this issue later in this section.

**Theorem 15** (Margulis/Fisher section 3.c in [BFGM]) *Let  $G$  be a locally compact second countable group. Assume  $G$  has property (T). Then the fixed-point spectrum  $\mathcal{F}_{L^\infty(0,1)}(G)$  contains a neighborhood of 2.*

We now sketch the proof of this theorem as it is given in section 3.c in [BFGM], since we will need some variation of it in the sequel.

**Proof** Let  $Q \subset G$  be a compact generating set. The theorem is a consequence of the following claim.

Claim : there exists  $C > 0$ ,  $\epsilon > 0$  such that for all  $q \in (2 - \epsilon, 2 + \epsilon)$ , for all affine isometric action  $\alpha$  on  $L_q(0, 1)$ , and for all  $x \in L_q(0, 1)$ , there exists  $y \in L_q(0, 1)$  such that:

$$\begin{aligned} \|x - y\|_q &\leq C \text{diam}(\alpha(Q)x) \\ \text{diam}(\alpha(Q)x) &\leq \text{diam}(\alpha(Q)y). \end{aligned}$$

From the claim, it is very easy to show that  $G$  has property  $(F_{L_q(0,1)})$  (see [BFGM]).

To prove the claim, we assume the contrary and show that this contradicts the fact that  $G$  has property  $(T)$ . Hence we assume that there exists a sequence of reals  $(p_n)_n$  converging to 2, a sequence  $(x_n)_n$  with  $x_n \in L_{p_n}(0, 1)$ , and affine isometric actions  $\alpha_n = (\pi_n, b_n)$  on  $L_{p_n}(0, 1)$  such that  $\text{diam}(\alpha_n(Q)x_n) = 1$  and

$$\text{diam}(\alpha_n(Q)y) \geq 1/2 \text{ for all } y \in B(x_n, C \times n) \text{ (*)} .$$

Then we define a limit action  $\alpha$  on the ultraproduct affine space  $(x_n)_n + (L_{p_n}(0, 1))_{\mathcal{U}}$ , as recalled in the previous discussion. Since  $\lim_n p_n = 2$ , the limit space is an affine Hilbert space and condition (\*) implies that  $\alpha$  has no  $G$ -fixed-point, contradicting property  $(T)$ .

For completeness of the proof, we add a few remarks about the continuity of the limit action. As said before there is no reason for the limit action to be continuous (see the Example 16 below) when  $G$  is a non-discrete locally compact group.

Nevertheless, one can restrict the action  $\alpha$  to some well-chosen closed subspace  $\mathcal{H}_0$  of the Hilbert limit space, in order to have  $\alpha|_{\mathcal{H}_0}$  continuous. This method was used in section 4 of [CCS] and details can be found there. We only recall how the subspace  $\mathcal{H}_0$  is obtained.

For  $n \in \mathbb{N}$  and  $f \in C_c(G)$ , one can define an affine map  $\alpha_n(f) : L_{p_n} \rightarrow L_{p_n}$  by the formulae

$$\alpha_n(f)x = \int_G f(g)\alpha_n(g)x \, dg \text{ for all } x \in L_{p_n}.$$

Moreover, we have  $\alpha(f)(x_n)_\mathcal{U} \in \mathcal{H}$  for all  $(x_n)_\mathcal{U} \in \mathcal{H}$  and all  $f \in C_c(G)$ . Then take an approximate identity  $(f_k)_k$  in  $L_1(G)$  with supports in a common compact neighborhood of the identity  $K \subset G$ , and we define

$$\mathcal{H}_0 = \{ x \in \mathcal{H} \mid \lim_k \|\alpha(f_k)x - x\| \}.$$

One can choose  $K$  such that  $\mathcal{H}_0$  does not depend on the approximate identity we choose. Then one can show that  $\mathcal{H}_0$  is a  $\alpha(G)$ -invariant closed subspace of  $\mathcal{H}$ , and that the maps  $G \rightarrow \mathcal{H}_0$ ,  $g \mapsto \alpha(g)x$  are continuous for all  $x \in \mathcal{H}_0$ .  $\blacksquare$

The following example shows that the continuity of the limit action in the above proof does not hold in the whole limit space in general.

**Example 16** Let  $G$  be a non-discrete topological group. Let  $\mathcal{U}$  be an ultrafilter on  $\mathbb{N}$ . We define  $\mathcal{H} = (\mathbb{R})_\mathcal{U}$  to be the ultrapower of copies of  $\mathbb{R}$  along  $\mathcal{U}$ . On the  $n$ -th copy of  $\mathbb{R}$ , define the orthogonal representation  $\pi_n : G \rightarrow O(L_2(\mathbb{R}))$  by

$$\pi_n(a)f(x) = f(x + na) \text{ for all } n \in \mathbb{N}, a, x \in \mathbb{R} \text{ and } f \in L_2(\mathbb{R}).$$

Now take  $\pi$  the natural limit action (which acts by linear isometries) on  $\mathcal{H}$  associated to the actions  $\pi_n$ . Denote also by  $f$  the diagonal embedding of  $f := \chi_{[0,1]}$ , the indicator function of  $[0,1]$ , in  $\mathcal{H}$ . It is easily checked that, for all  $a \neq 0$ , we have

$$\|\pi(a)f - f\| = 2.$$

Hence the limit action  $\pi$  is not continuous on  $\mathcal{H}$ .

For countable groups  $G$ , one can show the openness of the fixed-point spectrum  $\mathcal{F}_{L^\infty(0,1)}(G)$  in  $[1, \infty[$ .

**Proposition 17** *Let  $G$  be a finitely generated group. Then  $\mathcal{F}_{L^\infty(0,1)}(G)$  is open in  $[1, \infty[$ .*

The proof of this result is also based on Margulis/Fisher construction. In this proof, two arguments require the group  $G$  to be countable : the continuity of the limit action, as discussed previously; and the restriction of an arbitrary  $L_p(Y, \nu)$  limit space on which  $G$  acts on, to the classical  $L_p(0,1)$  space.

**Proof** Let  $p \geq 1$  and assume that  $G$  has property  $(F_{L_p(0,1)})$ . The proof goes as the proof of Theorem 15, replacing 2 by  $p$ . To prove the claim, we

assume the contrary as previously, and we obtain an unbounded isometric affine action  $\alpha = (\pi, b)$  of  $G$  on a space isometrically isomorphic to  $L_p(Y, \nu)$  for some measure space  $(Y, \nu)$ . By Lemma 7.2 in [NP], there exists an isometric affine action  $\alpha' = (\pi', b')$  on  $L_p(0, 1)$  such that  $\|b(g)\|_p = \|b'(g)\|_p$  for all  $g \in G$ . Hence the action  $\alpha'$  is unbounded as well, and this contradicts property  $(F_{L_p(0,1)})$ . ■

## 5.2 Fixed-point spectrum associated to measure preserving ergodic actions on finite measure spaces

As a consequence of the following proposition, the fixed-point spectrum  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  is an interval when the measure  $\mu$  is finite, and the representation  $\pi$  is measure-preserving and ergodic.

**Proposition 18** *Let  $1 \leq p < q < r < \infty$ . Let  $(X, \mu)$  be a Borel standard measure space such that  $\mu$  is finite. Let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  be a (BL) measure-preserving ergodic representation. Assume that*

$$H^1(G, \pi^p) = \{0\},$$

and

$$H^1(G, \pi^r) = \{0\}.$$

Then we have also

$$H^1(G, \pi^q) = \{0\}.$$

The proof follows the same lines as the proof for actions on  $\ell_p$ .

**Proof** Notice that we have  $L_r(X, \mu) \subset L_q(X, \mu) \subset L_p(X, \mu)$ . Let  $b : G \rightarrow L_q(X, \mu)$ . Since  $\pi = \pi^p$  is measure-preserving,  $b$  defines a cocycle with values in  $L_p(X, \mu)$ . By assumption, there exists  $f \in L_p(X, \mu)$  such that

$$b(g) = f - \pi^p(g)f \text{ for all } g \in G.$$

Triangle inequalities imply the following inequalities for all  $g \in G$ :

$$|||f| - |\pi(g)f|||_q \leq \|b(g)\|_q < \infty.$$

Now we apply estimates (2) from section 2.2 with  $q < r$ , and  $|f|$  to obtain:

$$|||f|^{q/r} - |\pi(g)f|^{q/r}||_r \leq |||f| - |\pi(g)f|||_q^{q/r} < \infty.$$

Hence the left hand-side of the previous inequality defines a cocycle with values in  $L_r$ , which is a coboundary by assumption. Since the action of  $G$  on  $(X, \mu)$  is ergodic we can write:

$$|f|^{q/r} = c + h$$

where  $c$  is a constant function, and  $h \in L_r(X, \mu)'$ . Then we have  $|f|^{q/r} \in L_r(X, \mu)$ , that is  $f \in L_q(X, \mu)$ . ■

Now the proof of Theorem 3 is an easy consequence of Proposition 18.

**Proof of Theorem 3** Let  $\pi : G \rightarrow O(L_p(X, \mu))$  be a measure-preserving ergodic on  $(X, \mu)$  finite. By Proposition 18, the set  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  is empty or is an interval. When the fixed-point spectrum is non-empty, it is closed on the right by proposition 6.

If  $G$  has property (T), we know that the set  $\mathcal{F}(G)$  contains an interval of the form  $[1, q[$  for some  $q > 2$  (see Theorem 15 from section 5.1, and Theorem 1.3 in [BFGM]). Combined with our Proposition 18, the set  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  is an interval of the form  $[1, p_c]$  or  $[1, \infty[$  for some  $p_c > 2$ . Hence the theorem is proved. ■

When  $G$  does not have property (T) and  $p \geq 1$ , one can construct a (Gaussian) finite measure space  $(X, \mu)$  endowed with an action of  $G$  such that :

- the action of  $G$  on  $(X, \mu)$  is measure-preserving;
- the restriction of the associated representation  $\rho^p$  to  $L'_p(\rho^p)$  admits a sequence of almost invariant vectors.

Thus by the usual Guichardet argument, we have

$$H^1(G, \rho^p) \neq \{0\}.$$

In particular, we have  $\mathcal{F}_{L^\infty(X, \mu)}(G, \rho) = \emptyset$ .

The construction of  $(X, \mu)$  above is due to Connes and Weiss, and details are explained in [BHV] (Theorem 6.3.4). See also section 4.c in [BFGM], for the proof of the statement related to almost invariant vectors.

The openness of the set  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  would imply  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi) = [1, \infty[$ , when  $G$  has property (T). Unfortunately, we could not derive an analog of Proposition 17 for a fixed-point spectrum restricted to measure-preserving actions.

**Question 19** If  $\pi$  is a measure-preserving ergodic (BL) representation on a finite measure space  $(X, \mu)$ , and if  $G$  has property (T), do we always have  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi) = [1, \infty[$  ?

### 5.3 Some results for non-measure preserving actions under additional assumptions

For a non-measure preserving representation but under some boundedness assumptions, we can use a result from [BN] about cohomology of deformations to show that the spectrum  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi)$  is connected.

Let  $G$  be a topological group, and let  $(X, \mu)$  be a standard Borel measure space. Let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  ( $1 \leq p < \infty$ ) be a (BL) orthogonal representation. The use of the theorems from [BN] requires some additional assumptions on the group  $G$  and the representation  $\pi^p$  considered, namely :  $G$  is finitely presented, and the second cohomology  $H^2(G, \pi^p)$  is reduced. We refer to [BN] for the relevant definitions and discussion on these assumptions.

Assume that  $G$  is generated by a finite set  $S$ . A representation  $\rho : G \rightarrow B(L_p(X, \mu))$  of  $G$  into the group of bounded linear operators  $B(L_p(X, \mu))$  is said to be an  $\epsilon$ -deformation of  $\pi^p$  in the sense of [BN] if

$$\sup_{s \in S} \sup_{\|f\|_p=1} \|\pi^p(s)f - \rho(s)f\|_p \leq \epsilon.$$

For  $g \in G$ , denote by  $\frac{d(g*\mu)}{d\mu}$  the Radon-Nikodym derivative which appears in the Banach-Lamperti description of the isometry  $\pi^p(g)$ . The conjugate representations  $(\pi^q)_{q \geq 1}$  are a family of deformation of  $\pi^p$ , and their deformations can be controlled whenever the Radon-Nikodym derivative  $\frac{d(g*\mu)}{d\mu}$  is bounded for all  $g \in G$  (see Example 9 in [BN]). Then we have the following result.

**Proposition 20** *Let  $G$  be a finitely presented group with property (T). Let  $(X, \mu)$  be a standard Borel measure space, and let  $\pi^p : G \rightarrow O(L_p(X, \mu))$  ( $1 \leq p < \infty$ ) be a (BL) orthogonal representation. Assume that  $H^2(G, \pi^p)$  is reduced, and that  $\frac{d(g*\mu)}{d\mu} \in L^\infty(X, \mu)$  for all  $g \in G$ . Then  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  is open.*

*If moreover  $H^2(G, \pi)$  is reduced for all unitary representations  $\pi$  of  $G$ , then the fixed-point spectrum  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  is closed. Hence in that case,  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  is connected.*

**Proof** Fix a finitely generated set  $S$  of  $G$ . We have the following inequality

$$\|\pi^r(s)f - \pi^q(s)f\|_p \leq \|1 - (\frac{d(s * \mu)}{d\mu})^{1/q-1/p}\|_\infty \text{ for all } f \in S(L_p(X, \mu)), s \in S.$$

And the right hand-side of the inequality tends to 0 as  $q$  tends to  $r \geq 1$  since the Radon-Nikodym derivative  $\frac{d(s * \mu)}{d\mu}$  is bounded for all  $s \in S$ .

Notice that  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  is not empty since  $G$  has property (T). Let  $r \in \mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$ . Then by the first part of Theorem 4 in [BN], there exists  $\epsilon = \epsilon(G, \pi^p)$  such that  $H^1(G, \pi^q) = \{0\}$  whenever  $q$  is  $\epsilon$ -close to  $r$ . Hence the openness of  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  at  $r$  is clear from the inequalities above.

By the second part of Theorem 4 in [BN], when  $H^2(G, \pi)$  is reduced for all unitary representations of  $G$ , there exists  $\epsilon = \epsilon(G)$  independant of  $\pi^p$  such that  $H^1(G, \pi^q) = \{0\}$  whenever  $q$  is  $\epsilon$ -close to  $r$ . Then the set  $\mathcal{F}_{L^\infty(X, \mu)}(G, \pi^p)$  is closed at  $r$  as well. ■

As pointed out in [BN], automorphisms of thick buildings are examples of groups which have vanishing of higher cohomology with coefficients in unitary representations (see Theorem B in [DJ]). So our Proposition 20 applies to such groups and their (BL) representations whose Radon-Nikodym derivatives satisfy the boundedness assumption. The following corollary is a straightforward consequence of Theorem 15, Proposition 20, and the well-known fact that property (T) is equivalent to property  $(F_{L_p(0,1)})$  for all  $1 \leq p \leq 2$ .

**Corollary 21** *Let  $G$  be a finitely presented group with property (T). Assume that  $H^2(G, \pi)$  is reduced for all unitary representations  $\pi$  of  $G$ . Then the fixed-point spectrum  $\mathcal{F}_{L^\infty(0,1)}(G, \pi^p)$  is an interval of the form  $[1, p_c[$  or  $[1, p_c]$  for some  $2 < p_c < \infty$ .*

## References

- [BFGM] U. Bader, A. Furman, T. Gelander, and N. Monod. Property (T) and rigidity for actions on Banach spaces. *Acta Math.*, 198: 57–105, 2007.
- [BGM] U. Bader, T. Gelander and N. Monod. A fixed-point theorem for  $L^1$ -spaces. *Invent. math.*, 189, 143–148, 2012.

- [BN] U. Bader, P. W. Nowak. Cohomology of deformations. *arxiv*: 1401.5362, 2014.
- [BHV] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T)*, Cambridge Univ. Press, 2008.
- [BL] Y. Benyamini and J. Lindenstrauss. *Geometric Nonlinear Functional Analysis*, Vol.1 American Mathematical Society Colloquium Publications, 48, 2000.
- [BO] B. Bekka, and B. Olivier. On groups with property  $(T_{\ell_p})$ . *Journal Funct. Anal.*, 267, No 3, 643–659, 2014.
- [B] M. Bourdon. Cohomologie et actions isométriques propres sur les espaces  $L^p$ . To appear in *Geometry, topology and Dynamics*, Proceedings of the 2010 Bangalore conference.
- [BMV] M. Bourdon, F. Martin and A. Valette. Vanishing and non-vanishing for the first  $L^p$ -cohomology groups. *Comment. Math. Helv.*, 80, 377–389, 2005.
- [CCS] P.-A. Cherix, M. Cowling and B. Straub. Filter products of  $C_0$ -semigroups and ultraproduct representations for Lie groups. *J. Funct. Anal.*, 208, no. 1, 31–63, 2004.
- [CDH] I. Chatterji, C. Drutu and F. Haglund. Kazhdan and Haagerup properties from the median viewpoint. *Adv. Math.*, 225, no. 2, 882–921, 2010.
- [DJ] J. Dymara and T. Januszkiewicz. Cohomology of buildings and their automorphism groups. *Invent. Math.*, 150, no. 3, 579–627, 2002.
- [G] M. Gromov. Asymptotic Invariants of Infinite Groups. *London Math. Soc. Lecture Note Ser.*, vol. 182, Cambridge Univ. Press, Cambridge, 1993.
- [H] S. Heinrich. Ultraproducts in Banach spaces theory. *J. Reine Angew. Math.*, 313, 72–104, 1980.
- [LS] T. de Laat and M. de la Salle. Strong property  $(T)$  for higher rank simple Lie groups. *arxiv* 1401.3611, 2014.



- [M1] M. Mimura. Fixed point properties and second bounded cohomology of universal lattices on Banach spaces. *J. Reine Angew. Math.*, Vol. 2011, No. 653, 115–134, 2011.
- [M2] M. Mimura. Fixed point property for universal lattice on Schatten classes. *Proc. Amer. Math. Soc.*, 141, no. 1, 65–81, 2012.
- [NP] A. Naor and Y. Peres.  $L_p$ -compression, traveling salesmen, and stable walks. *Duke Math. J.*, 157(1), 53–108, 2011.
- [Ni] B. Nica. Proper isometric actions of hyperbolic groups on  $L^p$ -spaces. *Composition Mathematica*, vol. 149, issue 5, 773–792, 2013.
- [No] P.W. Nowak. Group actions on Banach spaces. *preprint* arXiv:1302.6609v2.
- [O] B. Olivier. Kazhdan’s property (T) with respect to non-commutative  $L_p$ -spaces. *Proc. Amer. Math. Soc.*, 140, no. 12, 4259–4269, 2012.
- [R] E. Ricard. Hölder estimates for the noncommutative Mazur maps. *arxiv*: 1407.8334, 2014.
- [Ser] J-P. Serre. *Arbres, amalgames,  $SL_2$* , Astérisque 46, Soc. Math. France, 1977.
- [Yu] G. Yu. Hyperbolic groups admit proper affine isometric actions on  $\ell_p$ -spaces. *Geom. Funct. Anal.*, 15, no. 5, 1144–1151, 2005.

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